Thermoconvective instability of paramagnetic fluids in a nonuniform magnetic field

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The effect of a static, nonuniform magnetic field on a laterally unbounded nonconducting paramagnetic fluid layer heated from below or above is studied using a linear stability analysis of the Navier-Stokes equations supplemented by Maxwell's equations and the appropriate magnetic body force. Buoyancy-driven convection can be controlled by subjecting the layer to a nonuniform magnetic field. Theoretical predictions agree with experimental observations. [S1063-651X(98)12505-9]

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I. INTRODUCTION

Recent experiments [1,2] show that a strong inhomogeneous static magnetic field can induce magnetothermal convection and can enhance or suppress buoyancy-driven convection in electrically nonconducting paramagnetic fluids, depending on the relative orientation of the field and temperature gradients. In these experiments, a paramagnetic solution of gadolinium nitrate is placed in a cylindrical cell closed with two horizontal plates that are maintained at different temperatures. The cell is placed in a nonuniform magnetic field produced by a superconducting electric coil capable of providing a maximum field-field gradient product $|B\partial B/\partial z| = 250\ \mathrm{T^2/m}$, where B is the magnitude of the magnetic induction and z the vertical coordinate. The nonuniform field exerts a magnetic body force on this electrically nonconducting solution. These experiments observe the following phenomena: (a) when the cell is heated from above (temperature difference $\Delta T<0$) and the magnetic force is upward, the measured Nusselt number remains unity for $|B\partial B/\partial z| \leq 5\ \mathrm{T^2/m}$ for $|\Delta T|$ up to 32 °C, indicating no convection. However, when $|B\partial B/\partial z| = 6\ \mathrm{T^2/m}$, the Nusselt number begins to increase for $|\Delta T|>2$ °C, indicating the onset of magnetothermal convection. For given ΔT, the larger the product $|B\partial B/\partial z|$, the larger the Nusselt number, indicating enhanced convection; (b) when the cell is heated from below and the magnetic force is downward, the Nusselt number increases with increasing $|B\partial B/\partial z|$ for given ΔT, indicating the enhancement of the buoyancy-driven convection; and (c) when the cell is heated from below and the magnetic force is upward, for given ΔT, the Nusselt number decreases with increasing $|B\partial B/\partial z|$ for $|B\partial B/\partial z| \leq 5\ \mathrm{T^2/m}$, indicating the partial suppression of the convection. When $|B\partial B/\partial z| = 6\ \mathrm{T^2/m}$, the Nusselt number remains unity for ΔT up to 5 °C, indicating that the convection is completely suppressed for $\Delta T \leq 5$ °C. When $B\partial B/\partial z = 15\ \mathrm{T^2/m}$, the Nusselt number remains unity for ΔT up to 30 °C. These experiments reveal that convection in paramagnetic fluids can be controlled by external inhomogeneous magnetic fields.

When a pure fluid is placed in a static magnetic field **H**, Landau and Lifshitz [3] calculate the volume forces on the fluid [Eq. (34.3) in Ref. [3] converted to SI units],

$$\mathbf{f} = -\nabla p_0 + \frac{1}{2}\nabla\left[H^2\rho\left(\frac{\partial\mu}{\partial\rho}\right)_T\right] - \frac{H^2}{2}\nabla\mu + \mu\mathbf{j}\times\mathbf{H}, \quad (1)$$

where p_0 is the pressure in the absence of the field, ρ the density of the fluid, T the temperature, μ the magnetic permeability of the fluid, and **j** the electric current density in the fluid. In this paper, we limit our consideration to electrically insulating fluids, i.e., $\mathbf{j}=0$ and, accordingly, the last term vanishes. We also limit our consideration to paramagnetic pure fluids, e.g., oxygen, and assume that the dissipative forces that occur in colloidal ferrofluids [4] are negligible. As $\mu=\mu_0(1+\chi)$, $\mathbf{M}=\chi\mathbf{H}$, and $\nabla\times\mathbf{H}=\mathbf{0}$, we can rewrite Eq. (1) as

$$\mathbf{f} = -\nabla p + \mu_0\mathbf{M}\cdot\nabla\mathbf{H}, \quad (2)$$

where $p = p_0 + \mu_0 H^2[\partial(\chi v)/\partial v]_T/2$ is the modified pressure, μ_0 the permeability of free space, $v=1/\rho$ the specific volume, χ the volumetric susceptibility of the fluid, and **M** the magnetization (the magnetic moment per unit volume). The modified pressure gradient term in Eq. (2) does not induce convection because it is irrotational. The last term in Eq. (2) is the Kelvin body force [5] $\mathbf{f}_m = \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}$, which arises from the interaction between the local magnetic field **H** within the fluid and the molecular magnetic moments. This force tends to move paramagnetic fluids toward regions of higher magnetic field. For typical paramagnetic fluids, the magnetic susceptibility satisfies Curie's law [6] $\chi = C\rho/T$, where C is a constant characteristic of the fluid. When a horizontal paramagnetic fluid layer heated from below or above is placed in a uniform oblique magnetic field, the imposed vertical thermal gradient induces a vertical gradient in the magnetic susceptibility, yielding a spatially nonuniform Kelvin body force. This thermal gradient induced magnetic body force tends to destabilize the layer. Our previous study [7] shows that longitudinal rolls with axes parallel to the horizontal component of the field are the rolls most unstable to convection. The corresponding critical Rayleigh number and critical wavelength for the onset of such rolls are less than the well-known Rayleigh-Bénard values in the absence of magnetic fields. Vertical fields maximize these deviations, which vanish for horizontal fields. Horizontal fields increase the critical Rayleigh number and the critical wavelength for all rolls except longitudinal rolls. We emphasize that these Kelvin force effects differ from the always-stabilizing effects of the $\mathbf{j}\times\mathbf{B}$ force on a layer of electrically conducting fluid [8].

The goal of this paper is to develop a theory of magnetothermal convection for nonconducting paramagnetic fluids in a realistic nonuniform magnetic field. We consider a horizontal paramagnetic fluid layer heated from above or below in the presence of an inhomogeneous magnetic field. In addition to the thermal gradient induced magnetic body force described above, the imposed field gradient directly yields a Kelvin body force on the fluid. The imposed thermal gradient renders the curl of this force nonzero through the temperature-dependent magnetic susceptibility. It is this rotational body force that is responsible for the phenomena observed in the experiments mentioned above. This force can be utilized to balance the gravitational body force within the fluid layer, and to enhance or to suppress the buoyancy-driven convection. It can also be utilized to promote convection when the layer is heated from above, where gravity stabilizes the layer.

In this paper, a linear stability analysis of a horizontal layer of pure paramagnetic fluid heated from below or above in the presence of a nonuniform magnetic field shows that oscillatory instability cannot occur and that convection in this layer can be controlled by the nonuniform field. In Sec. II, we outline the basic equations and boundary conditions, and present the static state solution. In Sec. III, we summarize the governing equations for the convective flow. In Sec. IV, we study the linear stability analysis of the layer in the presence of an inhomogeneous magnetic field, and outline the numerical method used to solve the marginal equations. We summarize the main results and draw conclusions in Sec. V.

II. EQUATIONS OF MOTION

For an incompressible, pure, paramagnetic fluid in the presence of a static, nonuniform magnetic field, the Navier-Stokes equation subject to Eq. (2) takes the form

$$\rho\,\frac{d\mathbf{V}}{dt}=\rho\mathbf{g}-\boldsymbol{\nabla}p+\rho\nu\nabla^2\mathbf{V}+\mu_0(\mathbf{M}\cdot\boldsymbol{\nabla})\mathbf{H}, \tag{3}$$

where t is time, $\mathbf{V}$ the fluid velocity, $d/dt=\partial/\partial t+\mathbf{V}\cdot\boldsymbol{\nabla}$ the material derivative, $\mathbf{g}$ the acceleration of gravity, and ν the kinematic viscosity. We assume the viscosity is isotropic and independent of the magnetic field.

Conservation of energy yields the temperature equation for an incompressible paramagnetic fluid [7]

$$\rho c_{p,\mathbf{H}}\,\frac{dT}{dt}-\mu_0\mathbf{M}\cdot\,\frac{d\mathbf{H}}{dt}=\kappa\nabla^2T+\Phi, \tag{4}$$

where $c_{p,\mathbf{H}}$ is the specific heat capacity at constant pressure and magnetic field, κ the thermal conductivity (assumed constant), and Φ the viscous dissipation.

For an incompressible fluid, the equation of continuity reduces to

$$\boldsymbol{\nabla}\cdot\mathbf{V}=0. \tag{5}$$

For an electrically nonconducting fluid, we write the Maxwell's equations

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0, \tag{6a}$$

$$\boldsymbol{\nabla}\times\mathbf{H}=0, \tag{6b}$$

where $\mathbf{B}\equiv\mu_0(\mathbf{M}+\mathbf{H})$ is the magnetic induction.

The density equation of state is linearized about an average temperature T_a

$$\rho=\rho_a[1-\alpha(T-T_a)], \tag{7}$$

where α is the thermal expansion coefficient. We also employ the Boussinesq approximation by allowing the density to change only in the large gravitational body force term.

In this paper, the coordinate system of the horizontal layer is defined by $|z|\leqslant d/2$ with $\hat{\mathbf{z}}$ up. We consider the fluid layer placed in an external nonuniform magnetic field with a constant field gradient,

$$\mathbf{H}^{\rm ext}=\mathbf{H}_0+(\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{H}^{\rm ext},\quad\text{or}\quad H_i^{\rm ext}=H_{0i}+H_{1ij}x_j \qquad (i,j=x,y,z), \tag{8}$$

where $H_{1ij}\equiv\partial H_i^{\rm ext}/\partial x_j$ are assumed constants, $a_ib_i\equiv\Sigma_ia_ib_i=a_xb_x+a_yb_y+a_zb_z$, and $\mathbf{x}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$ is the position vector. The vector $\mathbf{H}_0$ is the magnetic field at the origin ($\mathbf{x}=\mathbf{0}$). Equations (6a) and (6b) require that the tensor $\boldsymbol{\nabla}\mathbf{H}^{\rm ext}$ is symmetric and traceless. This assumed field may be thought of as the leading terms in a Taylor series expansion of a more general field. In any event, it provides a good approximation of the applied magnetic field in the region of the convection cell in the experiments reported in [1,2].

The magnetic equation of state is linearized about the temperature T_a and an average magnetic field, $\mathbf{H}_a=H_{ax}\hat{\mathbf{x}}+H_{ay}\hat{\mathbf{y}}+H_{az}\hat{\mathbf{z}}$, to become

$$M_i(H_i,T)=M_{ai}+\chi_a(H_i-H_{ai})-\frac{\chi_aH_{ai}}{T_a}\,(T-T_a) \qquad (i=x,y,z), \tag{9}$$

where $\chi_a=C\rho_a/T_a$ and $M_{ai}=\chi_aH_{ai}$.

Equations (6a) and (6b) require that the normal component of magnetic induction and the tangential component of magnetic field are continuous across the top and bottom boundaries,

$$[\hat{\mathbf{z}}\cdot\mathbf{B}]_-^+=0\quad\text{and}\quad[\hat{\mathbf{z}}\times\mathbf{H}]_-^+=\mathbf{0}\quad\text{at}\quad z=\pm d/2. \tag{10a}$$

Here, $[q]_-^+=\lim_{\epsilon\to0}(q|_{z=\pm d/2+\epsilon}-q|_{z=\pm d/2-\epsilon})$ is the difference between the values of a quantity q above and below the boundaries. Rigid boundary conditions require a vanishing velocity

$$\mathbf{V}=\mathbf{0} \quad \text{at} \quad z=\pm d/2, \tag{10b}$$

and the temperature is assumed constant on each boundary

$$T=\begin{cases} T_0 & \text{at} \quad z=d/2 \\ T_1 & \text{at} \quad z=-d/2. \end{cases} \tag{10c}$$

To find the pure thermal conduction state, we write

$$\mathbf{V}_s=\mathbf{0} \quad \text{and} \quad T_s=T_a-\beta z. \tag{11}$$

The boundary conditions on the temperature Eq. (10c) require $T_a=(T_0+T_1)/2$, the average temperature of the layer, and $\beta=(T_1-T_0)/d=\Delta T/d$, the temperature gradient. Note that $\Delta T>0$ when the layer is heated from below. To find the static state for the magnetic field, we write

$$H_{si}=H_{ai}+\gamma_{ij}x_j \quad (i,j=x,y,z), \tag{12}$$

where γ_{ij} is an undetermined tensor. The boundary conditions on the field and induction Eq. (10a) require

$$\mathbf{H}_a=\mathbf{H}_0, \quad \text{and} \quad \gamma_{ij}=H_{1ij}-\frac{\chi_a H_{0z}\beta}{T_a}\delta_{iz}\delta_{jz} \quad (i,j=x,y,z), \tag{13}$$

where δ_{ij} is the delta function ($\delta_{ij}=1$ for $i=j$, and $\delta_{ij}=0$ for $i\neq j$). In obtaining these results, we have used the fact that the typical value of magnetic susceptibility for paramagnetic fluids is $\chi<10^{-3}$. Because H_{1ij} and $\delta_{iz}\delta_{jz}$ are symmetric, so is the tensor γ_{ij}.

III. EQUATIONS FOR CONVECTIVE STATES

To derive the governing equations for convective states, we add perturbations to the static state and substitute this perturbed state into Eqs. (3)–(6) to yield the equations governing these perturbations. To write these equations in dimensionless form, we choose d, d^2/D_T, ΔT, and $H_d \equiv \chi_a \Delta T H_0/(1+\chi_a)T_a$ as the scales for length, time, temperature, and magnetic field, respectively. Here, $D_T \equiv \kappa/\rho_0 c_{p,\mathbf{H}}$ is the thermal diffusivity. Finally, we write the dimensionless governing equations for the convective states,

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p'+R\theta\hat{\mathbf{z}}+K(\sin^2\phi)\theta\hat{\mathbf{z}} -\mathbf{R}_m\theta+K(z-\theta)\hat{\mathbf{H}}_0\cdot\nabla\mathbf{h}+\nabla^2\mathbf{v}, \tag{14}$$

$$\left(\frac{\partial\theta}{\partial t}+\mathbf{v}\cdot\nabla\theta-\hat{\mathbf{z}}\cdot\mathbf{v}\right) -L_1\hat{\mathbf{H}}_0'\cdot\left[\frac{\partial\mathbf{h}}{\partial t}+\mathbf{v}\cdot\nabla(\mathbf{h}+\mathbf{H}^{\mathrm{ext}}/H_d)-L_2\hat{\mathbf{z}}\cdot\mathbf{v}\right]=\nabla^2\theta+\Phi', \tag{15}$$

$$\nabla\cdot\mathbf{h}-\hat{\mathbf{H}}_0\cdot\nabla\theta=0, \tag{16}$$

$$\nabla\cdot\mathbf{v}=0. \tag{17}$$

In these equations, the variables $\mathbf{v}$, θ, and $\mathbf{h}$ represent the dimensionless perturbations. Here, p' is a reduced pressure including magnetic contributions, $\hat{\mathbf{H}}_0=\mathbf{H}_0/H_0$ is the unit vector in the direction of $\mathbf{H}_0$, $\hat{\mathbf{H}}_0'\equiv\mathbf{H}^{\mathrm{ext}}/H_0$, ϕ the angle between $\mathbf{H}_0$ and the horizontal, and Φ' the dimensionless viscous dissipation.

Equation (14) involves four dimensionless parameters: the Prandtl number Pr, the Rayleigh number R, the Kelvin number K, and the vector control parameter $\mathbf{R}_m$:

$$\mathrm{Pr}=\frac{\nu}{D_T}, \quad R=\frac{\alpha g d^3\Delta T}{\nu D_T}, \quad K=\frac{\mu_0\chi_a^2\Delta T^2 d^2 H_0^2}{(1+\chi_a)\rho_a T_a^2\nu D_T}, \tag{18a}$$

$$\mathbf{R}_m=\frac{\mu_0\chi_a d^3\Delta T}{\rho_a T_a\nu D_T}\mathbf{H}_0\cdot\nabla\mathbf{H}^{\mathrm{ext}}. \tag{18b}$$

The two dimensionless parameters in Eq. (15) are $L_1=\mu_0\chi_a^2H_0^2/(1+\chi_a)T_a\rho_a c_{p,\mathbf{H}}$, and $L_2=(1+\chi_a)\sin\phi\leqslant(1+\chi_a)$. Here, we use the values of ρ_a and $c_{p,\mathbf{H}}$ for water to estimate the typical value for the geometry-independent parameter L_1. A typical value for the magnetic susceptibility of paramagnetic fluids is $\chi_a\sim10^{-3}$. For a magnetic induction $B_0=10$ T, we have $L_1\sim10^{-7}\ll1$ at room temperature. We also have $L_1\sim10^{-4}$ for gaseous oxygen at room temperature. Accordingly, the term involving L_1 in Eq. (15) will be neglected.

In the presence of a uniform oblique magnetic field ($K\neq0$, but $\mathbf{R}_m=\mathbf{0}$), our linear stability analysis [7] shows that longitudinal rolls with axes parallel to the horizontal component of the field are the rolls most unstable to convection, reflecting the broken rotational symmetry of the layer about the vertical due to the presence of the nonzero horizontal component of the field.

In the presence of an inhomogeneous magnetic field ($K\neq0$ and $\mathbf{R}_m\neq\mathbf{0}$), the vector parameter $\mathbf{R}_m$ in Eq. (14) measures the strength of the magnetic body force due to the applied field gradient. The combination of the vertical component of $\mathbf{R}_m$ with R in Eq. (14) shows that the gravitational effect on the convective flow can be balanced by this component of $\mathbf{R}_m$. Therefore, convection in nonconducting paramagnetic fluids can be controlled by an inhomogeneous magnetic field.

IV. LINEAR STABILITY ANALYSIS

To investigate the magnetothermal convective instability, we assume that the amplitudes of $\mathbf{v}$, θ, and $\mathbf{h}$ are infinitesimal so that all cross terms in Eqs. (14) and (15) can be neglected. In this paper, we consider the nonuniform magnetic field $\mathbf{H}^{\mathrm{ext}}=H_0\hat{\mathbf{z}}-H_1x\hat{\mathbf{x}}-H_1y\hat{\mathbf{y}}+2H_1z\hat{\mathbf{z}}$, where the two parameters H_0 and H_1 are constants. (The superconducting electric coil used in the experiments [1,2] produces this field in the central area near the ends of the coil.) This field yields the vector parameter $\mathbf{R}_m=R_m\hat{\mathbf{z}}$, where

$$R_m=\frac{\mu_0\chi_a d^3\Delta T}{\rho_a T_a\nu D_T}\left(H\frac{\partial H}{\partial z}\right)^{\mathrm{ext}}_{\mathbf{x}=\mathbf{0}}. \tag{19}$$

Taking the z component of the curl of the linearized Eq. (14) yields

$$\frac{1}{\mathrm{Pr}}\frac{\partial \zeta}{\partial t}=\nabla^2\zeta, \tag{20}$$

where $\zeta\equiv\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla}\times\mathbf{v})$ is the z component of the vorticity. Taking the z component of the curl curl of Eq. (14), we have

$$\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\nabla^2 w=\nabla^4 w+(R-R_m+K)\nabla_\perp^2\theta-K\nabla_\perp^2\frac{\partial\psi}{\partial z}, \tag{21}$$

where w is the z component of velocity $\mathbf{v}$ and $\nabla_\perp^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$. Equations (15) and (16) yield

$$\frac{\partial\theta}{\partial t}=\nabla^2\theta+w, \tag{22}$$

$$\nabla^2\psi-\frac{\partial\theta}{\partial z}=0. \tag{23}$$

Equations (20), (21), (22), and (23) govern the linearized convective flow. The unboundedness in the horizontal direction allows the perturbation wavelength to be chosen freely in this direction, whence

$$\begin{aligned}
\zeta&=\zeta(z)\exp(iq_x x+iq_y y+\sigma t),\\
w&=w(z)\exp(iq_x x+iq_y y+\sigma t),\\
\theta&=\theta(z)\exp(iq_x x+iq_y y+\sigma t),\\
\psi&=\psi(z)\exp(iq_x x+iq_y y+\sigma t),\\
\Psi&=\Psi(z)\exp(iq_x x+iq_y y+\sigma t),
\end{aligned} \tag{24}$$

where q_x is the x component of the dimensionless wave number, $\mathbf{q}=q_x\hat{\mathbf{x}}+q_y\hat{\mathbf{y}}$, of these perturbations, q_y its y component, and σ the growth rate. Here, Ψ represents the perturbations of the magnetic field outside the fluid layer induced by the convective motion of the fluid and satisfies Laplace's equation $\nabla^2\Psi=0$. These perturbations should vanish far away from the layer. Substituting Eq. (24) into the Laplace's equation yields $\Psi(z)=\Psi_+\exp(-qz)$ for $z>1/2$ and $\Psi(z)=\Psi_-\exp(qz)$ for $z<1/2$, where Ψ_+ and Ψ_- are two undetermined constants and $q=\sqrt{q_x^2+q_y^2}$ is the magnitude of the wave number $\mathbf{q}$.

Equations (10a), (10b), and (10c) yield the dimensionless boundary conditions

$$\zeta=w=dw/dz=\theta=0 \quad \text{at} \quad z=\pm 1/2, \tag{25a}$$

$$(1+\chi_a)\frac{d\psi}{dz}=\begin{cases}-q\psi, & z=1/2\\ q\psi, & z=-1/2.\end{cases} \tag{25b}$$

The general solution of Eq. (20) subject to Eqs. (25a) shows that any perturbation in the vertical component of the vorticity must decay in time. Thus, we set $\zeta=0$ in the instability analysis without loss of generality.

To study the oscillatory instability of these perturbations, we substitute Eq. (24) into Eqs. (21), (22), and (23) to yield a set of ordinary differential equations, which can be solved by the Galerkin method. We expand w according to

$$w(z)=\sum_m A_m F_m(z), \tag{26}$$

where the functions F_m are a complete set of orthonormal solutions of

$$\frac{d^4Fm}{dz^4}=\lambda_m^4 F_m \tag{27}$$

satisfying $F_m(z)=dF_m/dz=0$ at $z=\pm 1/2$. The functions F_m are divided into two classes: even functions C_m and odd functions S_m defined by

$$C_m(z)=\frac{\cosh(\lambda_m z)}{\cosh(\lambda_m/2)}-\frac{\cos(\lambda_m z)}{\cos(\lambda_m/2)}$$

and

$$S_m(z)=\frac{\sinh(\mu_m z)}{\sinh(\mu_m/2)}-\frac{\sin(\mu_m z)}{\sin(\mu_m/2)}.$$

These functions and their numerical eigenvalues λ_m and μ_m have been tabulated [8]. We also expand θ and ψ in a series of $F_m(z)$. A one-term approximation yields

$$A_1\sigma^2+A_2\sigma+A_3=0, \tag{28}$$

where

$$A_1=(q^2+C_{11})/\mathrm{Pr},$$

$$A_2=\lambda_1^4+2C_{11}q^2+q^4+A_1(q^2+C_{11}),$$

$$A_3=\mathrm{Pr}A_1(\lambda_1^4+2C_{11}q^2+q^4)-(R-R_m+K)q^2+Kq^2G_{11}/(S_{11}+q^2).$$

Here,

$$\lambda_1=4.730\,040\,74,$$

$$C_{11}\equiv-\langle C_1|C_1''\rangle=-\int_{-1/2}^{1/2}C_1(z)[d^2C_1(z)/dz^2]dz=12.302\,616\,19,$$

$$G_{11}\equiv\langle C_1|S_1'\rangle=3.342\,015\,57,$$

and

$$S_{11}\equiv-\langle S_1|S_1''\rangle=46.050\,122\,36.$$

The onset of neutral oscillatory instability requires an imaginary growth rate $\sigma=i\omega$. Since the functions A_1, A_2, and A_3 are all real, Eq. (28) can be satisfied for $\sigma=i\omega$ only if $A_2=0$. Since $A_2>0$ for all values of q, an oscillatory instability cannot occur in the one-term approximation. By setting σ

$=0$, $K=0$, and $R_m=0$, Eq. (28) yields an approximate marginal state $R_0(q)$ in the absence of magnetic fields, which has a minimum value $R_{0c}=1887$ located at $q_{0c}=3.21$. A numerical calculation involving 360 terms yields $R_{0c}=1708$ and $q_{0c}=3.12$, which agree with the well-known critical values $R_{0c}=1707.762$ and $q_{0c}=3.117$ for the onset of convection [8]. In the absence of magnetic fields, the critical Rayleigh number R_{0c} for the rigid boundary conditions is larger than the critical Rayleigh number $R_c=27\pi^4/4$ for the free boundary conditions [8], indicating that the rigid boundary conditions tend to stabilize the layer compared with the free boundary conditions. Since our one-term analysis of instabilities of Eqs. (21), (22), and (23) for the rigid boundary conditions is qualitatively equivalent to the analysis of instabilities of these equations for the free boundary conditions in which $C_1(z)$ is replaced by $\cos(\pi z)$, we conclude that an oscillatory instability cannot occur in this system. Thus we limit our consideration to a stationary instability.

For the marginal state, the perturbations neither grow nor decay with time. Setting $\partial/\partial t=0$ in Eqs. (21), (22), and (23) yields the governing equations for this state. We adopt the algorithm of Stiles and Kagan [9] to solve these equations numerically. Comparing with the previous algorithm of expanding all variables w, θ, and ψ in series of $F_m(z)$, this algorithm yields more rapid convergence for successive approximations. First, we still expand w according to Eq. (26). We then write

$$\theta=\sum_m A_m\theta_m \quad \text{and} \quad \psi=\sum_m A_m\psi_m . \tag{29}$$

We substitute Eqs. (24), (26), and (29) into Eqs. (22) and (23), and then solve these equations individually to obtain the general solutions for θ_m and ψ_m. We use the boundary conditions, Eqs. (25), to determine the coefficients involved in these general solutions. Finally, we have

$$\theta_m=2\gamma_m\cosh qz-(D^2+q^2)C_m(z)/\lambda_m^- ,$$

$$\begin{aligned}\psi_m=&u_m\sinh qz+\gamma_m z\cosh qz\\&-(\lambda_m^+ +2q^2D^2)DC_m(z)/(\lambda_m^-)^2 \quad \text{for } F_m=C_m ,\end{aligned} \tag{30a}$$

and

$$\theta_m=2\delta_m\sinh qz-(D^2+q^2)S_m(z)/\mu_m^- ,$$

$$\begin{aligned}\psi_m=&v_m\cosh qz+\delta_m z\sinh qz\\&-(\mu_m^+ +2q^2D^2)DS_m(z)/(\mu_m^-)^2 \quad \text{for } F_m=S_m ,\end{aligned} \tag{30b}$$

where

$$u_m=\frac{\lambda_m^2}{q\exp(q/2)(\lambda_m^-)^2}\left(\lambda_m^+ +2q^4+4q^3\lambda_m\tanh\frac{\lambda_m}{2}\right)-\frac{\gamma_m}{2},$$

$$v_m=\frac{\mu_m^2}{q\exp(q/2)(\mu_m^-)^2}\left(\mu_m^+ +2q^4+4q^3\mu_m\coth\frac{\mu_m}{2}\right)-\frac{\delta_m}{2},$$

$\lambda_m^+\equiv\lambda_m^4+q^4$, $\lambda_m^-\equiv\lambda_m^4-q^4$, $\mu_m^+\equiv\mu_m^4+q^4$, $\mu_m^-\equiv\mu_m^4-q^4$, $\gamma_m\equiv\lambda_m^2/\lambda_m^-\cosh(q/2)$, $\delta_m\equiv\mu_m^2/\mu_m^-\sinh(q/2)$, and $D\equiv d/dz$. In obtaining these results, we made use of the fact that $\chi_a\ll 1$ for paramagnetic fluids.

Substituting the general solutions Eq. (30a) into Eq. (21), multiplying by $C_n(z)$, and integrating over $[-1/2,1/2]$ yield

$$\sum_m A_m b_{mn}=0, \quad n=1,2,3,\ldots \tag{31a}$$

with

$$\begin{aligned}b_{mn}=&\left(\lambda_m^+ +\frac{q^4R_r}{\lambda_m^-}-\frac{q^4\lambda_m^+K}{(\lambda_m^-)^2}\right)\delta_{mn}\\&-\left(2-\frac{R_r}{\lambda_m^-}+\frac{2q^4K}{(\lambda_m^-)^2}\right)q^2\langle C_n|D^2C_m\rangle\\&+\frac{q^2\lambda_m^2K}{\exp(q/2)(\lambda_m^-)^2}\left(\lambda_m^+ +2q^4+4\lambda_m q^3\tanh\frac{\lambda_m}{2}\right)\\&\times\langle C_n|\cosh qz\rangle-\left[2R_r+\frac{1}{2}(2+q)K\right]\\&\times q^2\gamma_m\langle C_n|\cosh qz\rangle+q^3\gamma_mK\langle C_n|z\sinh qz\rangle,\end{aligned} \tag{32a}$$

where $R_r\equiv R-R_m$. Similarly, substituting the general solutions Eq. (30b) into Eq. (21), multiplying by $S_n(z)$, and integrating over $[-1/2,1/2]$ yield

$$\sum_m A_m b'_{mn}=0, \quad n=1,2,3,\ldots \tag{31b}$$

with

$$\begin{aligned}b'_{mn}=&\left(\mu_m^+ +\frac{q^4R_r}{\mu_m^-}-\frac{q^4\mu_m^+K}{(\mu_m^-)^2}\right)\delta_{mn}-\left(2-\frac{R_r}{\mu_m^-}+\frac{2q^4K}{(\mu_m^-)^2}\right)\\&\times q^2\langle S_n|D^2S_m\rangle+\frac{q^2\mu_m^2K}{\exp(q/2)(\mu_m^-)^2}\left(\mu_m^+ +2q^4\right.\\&\left.+4\mu_m q^3\coth\frac{\mu_m}{2}\right)\langle S_n|\sinh qz\rangle\\&-\left[2R_r+\frac{1}{2}(2+q)K\right]q^2\delta_m\langle S_n|\sinh qz\rangle\\&+q^3\delta_mK\langle S_n|z\cosh qz\rangle.\end{aligned} \tag{32b}$$

Equations (31a) and (32a) govern the marginal state for the onset of convection with an even parity for w and θ, whereas Eqs. (31b) and (32b) govern the state with an odd solution for w and θ.

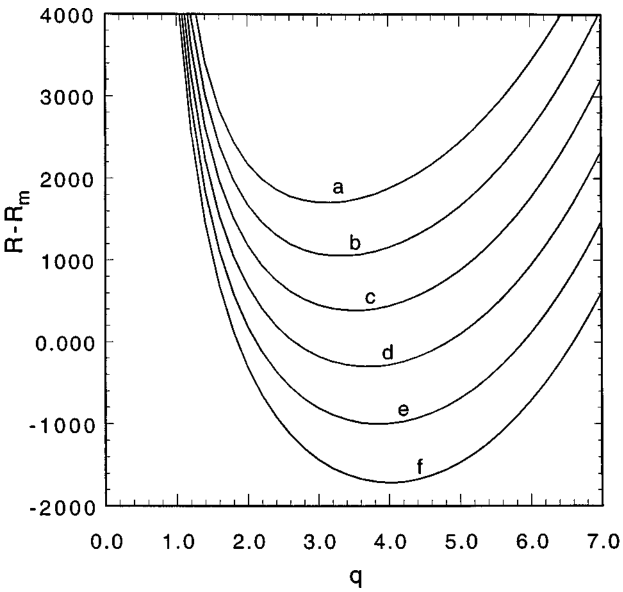

FIG. 1. Convective stability diagram for a horizontal layer of paramagnetic fluid in the presence of a nonuniform magnetic field. Shown are the marginal states for the reduced Rayleigh number $R_r(q) \equiv R - R_m$ for the onset of convection for $K=0$ (trace a), 1000 (trace b), 2000 (trace c), 3000 (trace d), 4000 (trace e), and 5000 (trace f) as a function of the wave number q.

V. RESULTS AND CONCLUSIONS

A nontrivial solution for the even case requires a vanishing determinant of the coefficient matrix in Eq. (31a), yielding the generalized marginal condition relating the reduced Rayleigh number $R_r = R - R_m$, the Kelvin number K, and the dimensionless wave number q. To obtain this condition numerically, we truncate the infinite series in Eq. (31a) to a finite number N of terms. For given values of N, q, and K, we adjust R_r numerically until the determinant vanishes. This procedure yields the marginal state for the reduced Rayleigh number $R_r = R_r(q,K)$, which can be minimized with respect to q to obtain the critical condition for R_r for given K.

For the classical, nonmagnetic case with $K=0$ and $R_m=0$ this procedure yields the known marginal Rayleigh number $R=R_0(q)$ in the absence of magnetic fields. Minimizing $R_0(q)$ with respect to q for $N=5$, 10, and 15 yields the successive estimates $R_{0c}=1707.784$, 1707.763, and 1707.762, with $q_{0c}=3.116$ in each case. These rapidly converge to the well-known critical values [8] $R_{0c}=1707.762$ and $q_{0c}=3.117$. Figure 1 shows $R_0(q)$ (trace a in Fig. 1) for the 15-term truncation, which is used henceforth. A static fluid layer in the absence of magnetic fields is stable to convective perturbations for Rayleigh numbers $R<R_{0c}$, above which gravitational buoyancy destabilizes a band of wave numbers centered approximately on q_{0c}.

For the onset of convection for the odd solution in the absence of magnetic fields, a numerical calculation of Eqs. (31b) and (32b) for the 15-term truncation yields the corresponding critical values $R_{0c}=17610.40$ and $q_{0c}=5.365$, consistent with their true values [8] $R_{0c}=17610.39$ and $q_{0c}=5.365$. Comparing with the critical Rayleigh number ($R_{0c}=1707.762$) for the onset of convection with an even solution, the large critical Rayleigh number for the odd solution implies that only the even solution is permitted when $R<17\,610$. Therefore, we limit our consideration to the marginal state governed by Eqs. (31a) and (32a).

TABLE I. Summary of results.

Case	ΔT	$\left(H\frac{\partial H}{\partial z}\right)_0^{\text{ext}}$	Kelvin force	R_m	Result
1	+	+	↑	+	Rayleigh-Bénard convection is inhibited
2	+	−	↓	−	Rayleigh-Bénard convection is promoted
3	−	−	↓	+	no convection
4	−	+	↑	−	magnetothermal convection is possible

Equations (31a) and (32a) govern the generalized marginal condition for the onset of convection of a horizontal layer of paramagnetic fluid in the presence of a nonuniform magnetic field. The Kelvin number K in these equations represents the uniform vertical field effect on convection, and R_m the effect due to the field gradient. Our numerical calculation yields these marginal conditions for the reduced Rayleigh number R_r as a function of the wave number q for given Kelvin numbers $K=0$, 1000, 2000, 3000, 4000, and 5000 (see Fig. 1). In the presence of a uniform vertical magnetic field ($K>0$ and $R_m=0$), Fig. 1 shows that the critical Rayleigh number R_c for the onset of convection is smaller than the classical nonmagnetic critical Rayleigh number $R_{0c}=1707.762$, indicating the enhancing effect of the field on convection. Figure 1 also shows that the larger the Kelvin number K, the smaller the critical Rayleigh number R_c, indicating the stronger the enhancing effect of the field on convection. Since $K\geqslant 0$, uniform vertical magnetic fields always promote convection, consistent with our previous analysis [7] and in contrast to the result for electrically conducting fluids [8]. In the absence of gravity ($R=0$), magnetic convection sets in when $K\geqslant K_{0c}=2568.476$ for uniform vertical fields.

In the presence of a nonuniform magnetic field ($K>0$ and $R_m\neq 0$), Fig. 1 shows that the effect on convection due to the field gradient depends on the sign of the magnetic control parameter R_m. A negative R_m will promote convection, and a positive R_m will inhibit convection. In this paper as well as in the experiments [1,2],

$$R_m = \frac{\mu_0 \chi_a d^3 \Delta T}{\rho_a T_a \nu D_T}\left(H\,\frac{\partial H}{\partial z}\right)_{\mathbf{x}=\mathbf{0}}^{\text{ext}}.$$

For paramagnetic fluids, all of the material properties are positive, so the sign of R_m is determined by the signs of ΔT and $(H\partial H/\partial z)_0^{\text{ext}}$. The four possible cases are summarized in Table I. In cases 1 and 2, the temperature difference ΔT is positive, indicating that the layer is heated from below. Gravity induces a gravitational buoyancy force that tends to destabilize the layer. In the absence of magnetic fields, Rayleigh-Bénard convection sets in for $R>R_{0c}$. In the presence of a nonuniform magnetic field, we see that an upward Kelvin force inhibits convection (case 1), whereas a down-

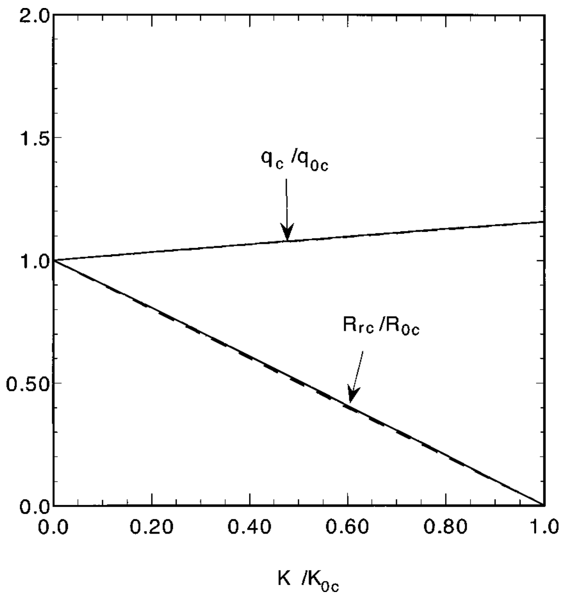

FIG. 2. Convective threshold for the ratio R_{rc}/R_{0c} of the critical reduced Rayleigh number R_{rc} from our simulations (dark trace) to the classical nonmagnetic critical Rayleigh number $R_{0c}=1707.762$ as a function of the ratio K/K_{0c} of the Kelvin number to the critical Kelvin number $K_{0c}=2568.476$ for uniform vertical fields and no gravity. Also shown is the threshold for the ratio q_c/q_{0c} of the critical wave number q_c to the classical nonmagnetic critical wave number $q_{0c}=3.116$ as a function of the ratio K/K_{0c}. Dashed traces give approximate linear results from Eqs. (33) and (34).

ward Kelvin force enhances convection (case 2). In cases 3 and 4, the layer is heated from above, and gravity tends to stabilize the layer. Table I shows that a downward Kelvin force enhances this stability (case 3), and there is no convection. However, an upward Kelvin force induces a magnetic buoyancy force that tends to destabilize the layer (case 4). Magnetothermal convection sets in when the reduced Rayleigh number $R_r > R_{rc}$, which is determined by minimizing the corresponding marginal $R_r(q,K)$ for given K with respect to the wave number q. The trends of cases 1, 2, and 4 are consistent with the experimental observations [1,2].

For a given Kelvin number K, minimizing the corresponding marginal value of the reduced Rayleigh number R_r with respect to the wave number q yields the critical reduced Rayleigh number R_{rc} and the critical wave number q_c. Figure 2 shows these critical numbers R_{rc} and q_c (solid traces) versus the Kelvin number K, scaled respectively by the values $R_{0c}=1707.762$, $q_{0c}=3.116$, and $K_{0c}=2568.476$. All points below the trace of R_{rc}/R_{0c} are stable to convection, whereas all points above this trace are unstable. The critical reduced Rayleigh number R_{rc} for the onset of convection decreases as the Kelvin number K increases, indicating the promoting effect of uniform vertical fields on convection. However, the critical wave number q_c for the onset of convection increases with increasing Kelvin number K, indicating that uniform vertical magnetic fields tend to narrow the pattern size of convection.

An approximate linear relation between R_{rc}/R_{0c} and K/K_{0c} follows from fits to the endpoints in Fig. 2;

$$\frac{R_{rc}}{R_{0c}}+\frac{K}{K_{0c}}=1, \tag{33}$$

where $R_{0c}=1707.762$ and $K_{0c}=2568.476$ as before. In the presence of a nonuniform magnetic field, Eq. (33) yields the critical reduced Rayleigh number R_{rc} for the onset of convection. For $0<R_{rc}/R_{0c}<1$, this relation conservatively gives K to within 1% over the entire parameter range (see the dashed trace in Fig. 2 for R_{rc}/R_{0c}). This relation reduces to our previous result [7] and the result obtained by Finlayson [10] for a uniform vertical field acting on a ferrofluid layer. Equation (33) provides a general condition for the field-field gradient product to balance the gravitational effect and hence to control convection in such fluids. Applying Eq. (33) to the experiments [1,2] yields the required product $|B\partial B/\partial z| = 5.3\ \mathrm{T}^2/\mathrm{m}$ to offset the effect of gravity in these experiments [11]. This value agrees well with the experimental measurements in cases (a) and (c) as described in the Introduction. However, we note that the experiments used a gadolinium nitrate solution but our theory assumes a pure paramagnetic fluid. Our analysis ignores Soret effects, which we cannot estimate due to lack of material properties. The agreement between the theory and experiments suggests that these effects are negligible.

An approximate linear relation between q_c/q_{0c} and K/K_{0c} also follows from fits to the end points in Fig. 2:

$$\frac{q_c}{q_{0c}}=1+0.159\,\frac{K}{K_{0c}}, \tag{34}$$

where $q_{0c}=3.116$, the critical wave number in the absence of magnetic fields. This relation gives the critical wave number q_c for the onset of convection to within 0.3% of its true value for $0<K/K_{0c}<1$. Equation (34) shows that the critical wave number $q_c>q_{0c}$ for $K>0$. Thus, the critical wavelength $\lambda_c=2\pi/q_c$ is smaller than the critical wavelength $\lambda_{0c}=2\pi/q_{0c}$ in the absence of magnetic fields, indicating that uniform vertical fields tend to narrow the pattern size of convection. Equation (34) also shows that this effect is independent of the magnetic control parameter R_m; and therefore, it is independent of the field gradient.

In conclusion, our linear stability analysis of a horizontal paramagnetic fluid layer heated from below or above predicts that convection in this layer can be controlled by a nonuniform magnetic field. The gravitational buoyancy in such a fluid layer due to thermal expansion can be balanced by the Kelvin body force due to the external field gradient. Thus, nonuniform magnetic fields can be used to enhance or to suppress the gravitational effect in terrestrial experiments and to control the flow of nonconducting paramagnetic fluids in microgravity environments. They can also be used to increase the efficiency of heat-transfer devices. They might also be used to control microstructures in crystal growth from paramagnetic liquids.

ACKNOWLEDGMENT

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